Pricing an exotic ADR Option

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This exercise culminated from the final project of the Structured Product and Hybrid Securities I enrolled in Fall 2018. The instructor was Professor Alireza Javeheri. I am intending to price the following pay-off. As of January 9th 2019, this project is incomplete.

$$E\left[e^{-\int_t^T r_d(u)du} \left(S_T^f X_T - K\right)^+ 1_{\left\{L_{T_1,T_1+\frac{1}{4}}(T_1)<B\right\}} \middle| \mathcal{F}_t \right]$$

Where $r_d(t)$ is the OIS rate, S_T^f is the Stoxx50E Index, X_T is the USD-EUR exchange rate, K is the strike, $L_{T_1,T_1+\frac{1}{4}}(t)$ is the LIBOR rate between T_1 and $T_1+\frac{1}{4}$ $(t \le T_1 < T_1 + \frac{1}{4} < T)$ and B is the barrier level.

I consider a multi-curve approach to model forward LIBOR and OIS curve. I assume Forward LIBOR follows a shifted LMM, and OIS evolves according to a Hull-White model. I calibrate the shifted LMM model using Black-76 Formula and OIS parameters by minimizing the volatility of the LIBOR-OIS basis. Stoxx50 Index is modeled by the two factor Bergomi model. To be more precise, I apply the Bergomi model to Stoxx50 Index in USD by adjusting for the foreign exchange rate. Resulting process has an advantage of being a martingale under a domestic (US) risk neutral measure. For calibration of Stoxx50 Index, I adopt a version of the particle method suitable for SLV and stochastic short rates. This requires the Dupire local volatility, which I obtain from the SVI interpolation.

As a recap, the ADR stock follows the following stochastic process. $d\tilde{S}_t^f = \tilde{S}_t^f\left(r_d dt + (\sigma_X + \sigma_f)dW_t\right)$ where $\tilde{S}_t^f = S_t^f X_t$, σ_X is the volatility of foreign exchange rate X_t, and σ_f is the volatility of the foreign stock S_t^f.

1. Model specification

1.1. Shifted LIBOR Market Model (SLMM)

Firstly, we observe that $L_j(t)$ is martingale under $\mathbb{Q}^{T_j}$ with $P(t,T_j)$ as the numeraire.

$$L_j(t) = \mathbb{E}^{T_j}\left[L(T_{j-1},T_j)\right] = \frac{1}{T_{j-1}-T_j}\mathbb{E}^{T_j}\left[\frac{P(T_{j-1},T_{j-1})}{P(T_{j-1},T_j)} - 1\right] = \frac{1}{T_{j-1}-T_j}\frac{P(t,T_{j-1})}{P(t,T_j)} - 1$$
$$= L(t,T_{j-1},T_j)$$

Thus, it makes sense to have a driftless shifted LMM dynamics:

$$d\tilde{L}_j(t) = \tilde{L}_j(t)\sigma_j(t)dW_j(t)$$

where $\tilde{L}_j(t) = L_j(t) + \alpha_j$, $dW_j(t)$ is $\mathbb{Q}^{T_j}$-Brownian Motion. α_j adjusts for skewness.
Since I utilize a single factor SLLM, $d\langle W_j, W_i\rangle_t = dt$.

The drift of $L_j(t)$ under $\mathbb{Q}$ is given by $\frac{d\left\langle L_j(\cdot),\ln\left(B(\cdot)/P(\cdot,T_j)\right)\right\rangle_t}{dt}$. It could be derived from the Girsanov Theorem. $\frac{d\mathbb{Q}^{T_j}}{d\mathbb{Q}} = \frac{P(T_j,T_j)/P(0,T_f)}{B(T_j)/B(0)}$ with $dW^{\mathbb{Q}}(t) = \varphi_t dt + dW_j(t)$

φ_t is the volatility of $\ln\left(B(t)/P(t,T_j)\right)$. Then,

$$d\tilde{L}_j(t) = -\varphi\tilde{L}_j(t)\sigma_j(t)dt + \sigma_j(t)\tilde{L}_j(t)dW^{\mathbb{Q}}(t)$$

which is consistent with $\frac{d\left\langle L_j(\cdot),\ln\left(B(\cdot)/P(\cdot,T_j)\right)\right\rangle_t}{dt}$

1.2. OIS dynamics

I assume that an instantaneous OIS rate $r(t)$ follows the Hull-White model.

$$r(t) = x(t) + \alpha(t)$$

$$dx(t) = -ax(t)dt + \sigma_r(t)dZ(t), x(0) = 0$$

Z is $\mathbb{Q}$-Brownian Motion

This gives

$$x(t) = e^{-at}\int_0^t e^{au}\sigma_r(u)dZ(u)$$

And,

$$x(T) = x(t)e^{-a(T-t)} + e^{-aT}\int_t^T e^{au}\sigma(u)dZ(u)$$

Thus, $\mathbb{E}[x(T)|\mathcal{F}_t] = x(t)e^{-a(T-t)}$ and $\mathbb{V}[x(T)|\mathcal{F}_t] = e^{-2aT}\int_t^T e^{2au}\sigma^2(u)du$

Also, integrating $dx(t)$ from T to t gives,

$$\int_t^T x(u)du = \frac{1}{-a}\left(x(T) - x(t)\right) + \frac{1}{a}\int_t^T \sigma(u)dZ(u)$$

Substituting in $x(T)$, we have

$$\int_t^T x(u)du = B(t,T)x(t) + \int_t^T \sigma(u)B(u,T)dZ(u)$$

where, $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$.

Therefore,

$\mathbb{E}\left[\int_t^T x(u)du\,|\mathcal{F}_t\right] = B(t,T)x(t)$ and $\mathbb{V}\left[\int_t^T x(u)du\middle|\mathcal{F}_t\right] = \int_t^T \sigma^2(u)B(u,T)^2du$

And, using Gaussian property, we have

$$\mathbb{E}\left[e^{-\int_t^T x(u)du}|\mathcal{F}_t\right] = \exp\left(-B(t,T)x(t) + \frac{1}{2}\int_t^T \sigma^2(u)B(u,T)^2du\right)$$
$$= e^{-B(t,T)x(t)}A(t,T)$$

Then,

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T x(u)du}|\mathcal{F}_t\right]e^{-\int_t^T \alpha(u)du}$$
$$= A(t,T)e^{-B(t,T)x(t)}e^{-\int_t^T \alpha(u)du}$$

This gives,

$$\int_t^T \alpha(u)du = -B(t,T)x(t) + \ln\frac{A(t,T)}{P(t,T)}$$
$$\int_0^T \alpha(u)du = \ln\frac{A(0,T)}{P(0,T)}$$

Thus,

$$P(t,T) = \frac{A(t,T)A(0,t)}{A(0,T)}\frac{P(0,T)}{P(0,t)}e^{-B(t,T)x(t)}$$

To model LIBOR-OIS Basis, we want to model the OIS forward rate.
The forward rate is defined as

$$F_j(t) = \frac{1}{\tau_j}\left(\frac{P(t,T_{j-1})}{P(t,T_j)} - 1\right) \text{ where } \tau_j = T_j - T_{j-1}$$

From the formula for $P(t,T)$ we obtain

$$\frac{P(t,T_{j-1})}{P(t,T_j)} = \frac{P(0,T_{j-1})}{P(0,T_j)}\frac{A(0,T_j)}{A(0,T_{j-1})}\frac{1}{A(T_{j-1},T_j)}e^{\left(B(t,T_j)-B(t,T_{j-1})\right)x(t)}$$

$$d\frac{P(t,T_{j-1})}{P(t,T_j)} = \frac{P(t,T_{j-1})}{P(t,T_j)}\left(B(t,T_j) - B(t,T_{j-1})\right)\sigma_r(t)dZ_j(t)$$

where, $Z_j(t)$ is $\mathbb{Q}^{T_j}$-Brownian Motion. We have zero drift by the choice of the numeraire.
Thus, combining the result,

$$dF_j(t) = \left(F_j(t) + \frac{1}{\tau_j}\right)\left(B(t,T_j) - B(t,T_{j-1})\right)\sigma_r(t)dZ_j(t)$$

And, $d\langle W_j, Z_j\rangle_t = \rho_{L,F}dt$

1.3. LIBOR-OIS Basis

The multiplicative LIBOR-OIS Basis is defined as

$$B_j^M(t) = \frac{1}{\tau_j}\left(\frac{L_j(t) + \frac{1}{\tau_j}}{F_j(t) + \frac{1}{\tau_j}} - 1\right)$$

Clearly, this definition is derived from following relationship between variables.

$$1 + \tau_j L_j(t) = \left(1 + \tau_j F_j(t)\right)\left(1 + \tau_j B_j^M(t)\right)$$

$$dB_j^M(t) = \frac{1}{\tau_j}\left(\frac{d\widetilde{L_j}(t)}{F_j + \frac{1}{\tau_j}} - \frac{\widetilde{L_j}(t) + \frac{1}{\tau_j}}{\left(F_j(t) + \frac{1}{\tau_j}\right)^2}dF_j(t) + \cdots dt\right)$$

$$= \frac{\left(B(t,T_j) - B(t,T_{j-1})\right)\left(L_j(t) + \frac{1}{\tau_j}\right)}{1 + \tau_j F_j(t)}$$

$$*\left(\frac{\sigma_j(t)\widetilde{L_j}(t)}{\left(B(t,T_j)-B(t,T_{j-1})\right)\left(L_j(t)+\frac{1}{\tau_j}\right)}dW_j(t)-\sigma_r(t)dZ_j+\cdots dt\right)$$

1.4. Bergomi two factor model

The Bergomi model presented in this section is the particular type that we discussed in the class. It falls into the class of Stochastic Local Volatility models that not only tries to capture stochastic nature of the volatility, but also ensures consistency with the Dupire local volatility. Unlike the model in the class, however, I assume all the stochastic characteristics are driven by $\mathbb{Q}^{T_j}$-Brownian Motions instead of $\mathbb{Q}$-Brownian Motions because we account for stochastic interest rates. The model is as following:

$$df(t)=\sqrt{\zeta_t^t}\sigma\left(t,P(t,T)^{-1}f(t)\right)f(t)dW_T^S(t)+B(t,T)\sigma_r dZ_T(t)$$
$$dX_t^1=-k_1X_t^1dt+dW_t^1,$$
$$dX_t^2=-k_2X_t^2dt+dW_t^2$$
$$\zeta_t^t=\zeta_0^t g(t,X_t^1,X_t^2)$$
$$g(t,x_1,x_2)=e^{2\nu\alpha_\theta[(1-\theta)x_1+\theta x_2]-\frac{(2\nu)^2}{2}\chi(t,t)}$$
$$\chi(t,T)=\alpha_\theta\left[\begin{array}{c}(1-\theta)^2e^{-2k_1(T-t)}\frac{(1-e^{-2k_1t})}{2k_1}+\theta^2e^{-2k_2(T-t)}\frac{1-e^{-2k_2t}}{2k_2}\\ +2\theta(1-\theta)\rho_{12}e^{-(k_1+k_2)(T-t)}\frac{(1-e^{-(k_1+k_2)t})}{k_1+k_2}\end{array}\right]$$
$$\chi(t,t)=\alpha_\theta\left[(1-\theta)^2\frac{(1-e^{-2k_1t})}{2k_1}+\theta^2\frac{1-e^{-2k_2t}}{2k_2}+2\theta(1-\theta)\rho_{12}\frac{(1-e^{-(k_1+k_2)t})}{k_1+k_2}\right]$$
$$\alpha_\theta=1/\sqrt{(1-\theta)^2+\theta^2+2\rho_{12}\theta(1-\theta)}$$
$$\theta\in[0,1],\ X_{t=0}^1=0,\ X_{t=0}^2=0,\ d\left\langle W_j^S,Z_j\right\rangle_t=\rho_{s,P}dt$$

2. Calibration

2.1. Calibration procedures for SLLM

The calibration procedure is standard. We use Black's formula for caplets. The approach is as following:

$$capl_j^B(t)=\mathbb{E}\left[e^{-\int_t^{T_j}r(u)du}\tau_j\max\left(L_j(T_{j-1})-R,0\right)\middle|\mathcal{F}_t\right]$$

Since $L_j(t)=\frac{1}{\tau_j}\left(\frac{P(t,T_{j-1})}{P(t,T_j)}-1\right)$ is $\mathbb{Q}^{T_j}$-Martingale because $P(t,T_j)$ is the numeraire. Thus,

$$capl_j^B(t)=\tau_jP(t,T_j)\mathbb{E}^{T_j}\left[\max\left(L_j(T_{j-1})-R,0\right)\mid\mathcal{F}_t\right]$$

Then, Black-Scholes formula gives

$$capl_j^B(t)=\tau_jP(t,T_j)\left(L_j(t)N(d_1)-RN(d_2)\right)$$

where, $d_1=\frac{\ln(L_j(t)/R)+\frac{1}{2}\sigma_j^2(T_j-t)}{\sigma_j\sqrt{T_j-t}}$ and $d_2=d_1-\sigma_j\sqrt{T_j-t}$

Calibration for the SLLM is carried out by equating the ATM SLLM with the ATM LLM.

$$\tau_j P(0,T_j)\left(2\tilde{L_j}(0)N\left(\frac{1}{2}\sigma_j\sqrt{T_j-t}\right)-1\right)=\tau_j P(0,T_j)\left(2L_j(0)N\left(\frac{1}{2}\sigma_{j,LLM}^{ATM}\sqrt{T_j-t}\right)-1\right)$$

Using the Taylor approximation to the normal distribution, $N(x)\approx\frac{1}{2}+\frac{x}{\sqrt{2\pi}}$, we have

$$\sigma_j\approx\frac{\sigma_{j,LMM}^{ATM}L_j(0)}{L_j(0)+\alpha_j}$$

To simplify the computation further for this project, I assume $\alpha_j=\frac{1}{\tau_j}$.

$\sigma_{j,LMM}^{ATM}$ is available on Bloomberg.

However, the problem of this project is a little bit more complicated in the sense that we want to simulate cashflows using $\mathbb{Q}^T$, but barrier condition is based on $\mathbb{Q}^{T_1}$ where $T>T_1$.

$$Q_i^{i-1}(t)\equiv\frac{d\mathbb{Q}^{i-1}}{d\mathbb{Q}^i}\bigg|\,\mathcal{F}_t=\frac{P(t,T_{i-1})/P(0,T_{i-1})}{P(t,T_i)/P(0,T_i)}$$

This implies,

$$Q_i^{i-1}(t)=\frac{P(0,T_i)}{P(0,T_{i-1})}\left(1+\tau_iL_i(t)\right)$$

$$dQ_i^{i-1}(t)=\frac{P(0,T_i)}{P(0,T_{i-1})}\tau_iL_i(t)\sigma_i(t)dW_i(t)$$

$$=Q_i^{i-1}\frac{\tau_iL_i(t)}{\left(1+\tau_iL_i(t)\right)}\sigma_i(t)dW_i(t)$$

By the definition of $Q_i^{i-1}(t)$.

The Girsanov theorem suggests $dW_i(t)=\frac{\tau_iL_i(t)}{(1+\tau_iL_i(t))}\sigma_i(t)dt+dW_{i-1}(t)$

$$dW_T(t)=\sum_{T_1<i\le T}\frac{\tau_iL_i(t)}{\left(1+\tau_iL_i(t)\right)}\sigma_i(t)dt+dW_{T_1}(t)$$

Thus, $\tilde{L}_{T_1}$ should be simulated using

$$d\tilde{L}_{T_1}(t)=-\sum_{T_1<i\le T}\frac{\tau_i\tilde{L}_i(t)\sigma_i(t)}{\left(1+\tau_i\tilde{L}_i(t)\right)}\tilde{L}_{T_1}(t)\sigma_{T_1}(t)\,dt+\tilde{L}_{T_1}(t)\sigma_{T_1}(t)dW_T(t)$$

Then,

$$\tilde{L}_{T_1}(t)=\tilde{L}_{T_1}(0)\exp\left(-\sum_{T_1<i\le T}\int_0^t\frac{\tau_i\tilde{L}_i(s)\sigma_i(s)}{\left(1+\tau_i\tilde{L}_i(s)\right)}\sigma_{T_1}(s)ds-\frac{1}{2}\int_0^t\sigma_{T_1}(s)ds\right.$$
$$\left.+\int_0^t\sigma_{T_1}(s)dW_T(s)\right)$$

And, for $T_1<i\le T$

$$\tilde{L}_i(t)=\tilde{L}_i(0)\exp\left(-\sum_{i<j\le T}\int_0^t\frac{\tau_j\tilde{L}_j(s)}{\left(1+\tau_j\tilde{L}_j(s)\right)}\sigma_j(s)\,ds-\frac{1}{2}\int_0^t\sigma_i(s)ds+\int_0^t\sigma_i(s)dW_T(s)\right)$$

2.2. Calibration procedures for OIS

We want to calibrate OIS curve by minimizing the instantaneous volatility of the multiplicative basis defined in 1.3.

$$\mathbb{V}\left[dB_j^M(t)\right] = \frac{\left(B(t,T_j) - B(t,T_{j-1})\right)^2 \left(L_j(t) + \frac{1}{\tau_j}\right)^2}{\left(1 + \tau_j F_j(t)\right)^2}$$

$$* \left(\frac{\sigma_j^2(t)\widetilde{L_j}^2(t)}{\left(B(t,T_j) - B(t,T_{j-1})\right)^2 \left(L_j(t) + \frac{1}{\tau_j}\right)^2} + \sigma_r^2(t) - \frac{2\rho_{L,F}\sigma_j(t)\sigma(t)\widetilde{L_j}(t)}{\left(B(t,T_j) - B(t,T_{j-1})\right)\left(L_j(t) + \frac{1}{\tau_j}\right)} \right) dt$$

Then, the first order and the second order conditions tell us that the instantaneous variance takes the minimum when

$$\sigma_r^*(t) = \frac{\rho_{L,F}\sigma_j(t)\widetilde{L_j}(t)}{\left(B(t,T_j) - B(t,T_{j-1})\right)\left(L_j(t) + \frac{1}{\tau_j}\right)}$$

Due to the complexity of this project, I will assume $\sigma_r^*(t) = \sigma_r$ is a constant and calibrate σ_r as:

$$\sigma_r = \frac{\rho_{L,F}\sigma_{T,LMM}^{ATM} L_T(0)}{\left(B(0,T) - B(0,T_1)\right)\left(L_T(0) + \frac{1}{T - T_1}\right)}$$

where T_1 is the trigger date for the barrier and T is the maturity.

2.3. Calibration procedures for implied volatility surface (SVI)

To calibrate the volatility surface of the Stoxx 50 Index options, I adopted the procedures outlined in "Quasi-Explicit Calibration of Gatheral's SVI model," by Zeliade Systems. SVI parametrizations is as following:

$$v(x) = \sigma_{BS}^2(x) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right)$$

Where $x = \ln(K/F_T)$.

Defining $\tilde{v} = Tv$, $y = \frac{x-m}{\sigma}$, $\tilde{v}(y) = aT + b\sigma T\left(\rho y + \sqrt{y^2+1}\right)$, $c = b\sigma T$, $d = \rho b \sigma T$, $\tilde{a} = aT$.

Then, $\tilde{v}(y) = \tilde{a} + dy + c\sqrt{y^2+1}$

Step 1 involves following optimization for each maturity.

$$\min_{\tilde{a},d,c} \sum_{i=1}^{n} \left(\tilde{a} + dy_i + c\sqrt{y_i^2 + 1} - \tilde{v}_i\right)^2$$

Subject to

$$\begin{cases} 0 \le c \le 4\sigma \\ |d| \le c \text{ and } |d| \le 4\sigma - c \\ 0 \le \tilde{a} \le \max_i \tilde{v}_i \end{cases}$$

With an initial choice of σ and m
Suppose the optimal solution of the problem is $(\tilde{a}^*, d^*, c^*)$

Step 2 is the following optimization problem

$$\min_{m,\sigma} \sum_{i=1}^{n} \left(\tilde{a}^* + d^* \frac{(x_i - m)}{\sigma} + c^* \sqrt{\left(\frac{x_i - m}{\sigma}\right)^2 + 1} - \tilde{v}_i \right)^2$$

Then, we can obtain the Dupire local volatility from the equation (2.19) of Stochastic Volatility Modeling by Lorenzo Bergomi.

2.4. Calibration procedures for the two factor Bergomi model and the generalized Dupire

Calibration is based on the static replication formula for variance futures using standard options. For example, VSTOXX futures with STOXX50E options or VIX futures with SP500 options.

Either (3.6) in Bergomi (2016) or (11.19) in Derman (2016) gives the static replication formula for an asset V.

$$\mathrm{V(S,T)} = V(K_0, T) + \left.\frac{\partial V(K,T)}{\partial K}\right|_{K=K_0} (S - K_0) + \int_0^{K_0} \frac{\partial^2 V(K,T)}{\partial K^2} \underbrace{(K-S)^+}_{P(S,t=T,K,T)} dK + \int_{K_0}^{\infty} \frac{\partial^2 V(K,T)}{\partial K^2} \underbrace{(S-K)^+}_{C(S,t=T,K,T)} dK$$

Let $V = S_T^2$. Then,

$$S_T^2 = K_0^2 + 2K_0(S_T - K_0) + \int_0^{K_0} 2\,(K - S_T)^+ dK + \int_{K_0}^{\infty} 2\,(S_T - K)^+ dK$$

Thus, when the underlying asset S_t is the VSTOXX futures contract $F_t^{T_i}$ that matures at T_i[1].
Taking $\mathbb{E}[\cdot\,|\mathcal{F}_t]$ on both sides gives the forward variance $\hat{\sigma}$,

$$\begin{aligned}\hat{\sigma}^2_{VST,T_i}(t) &\equiv \mathbb{E}\left[\left(F_{T_i}^{T_i}\right)^2 \middle| \mathcal{F}_t\right] \\ &= K_0^2 + 2K_0\left(F_t^{T_i} - K_0\right) + \int_0^{K_0} 2\mathbb{E}[(K - S_T)^+|\mathcal{F}_t]\, dK \\ &\quad + \int_{K_0}^{\infty} 2\,\mathbb{E}[(S_T - K)^+|\mathcal{F}_t] dK\end{aligned}$$

Then, $\hat{\sigma}^2_{VST,T_i}(t) = \left(F_t^{T_i}\right)^2 + \int_0^{F_t^{T_i}} 2\mathbb{E}[(K - S_T)^+|\mathcal{F}_t]\, dK + \int_{F_t^{T_i}}^{\infty} 2\,\mathbb{E}[(S_T - K)^+|\mathcal{F}_t] dK$

By letting $K_0 = F_t^{T_i}$.

[1] By the discussion in Appendix 5.1, $F_{T_i}^{T_i} \equiv \sqrt{\hat{\sigma}^2_{VST,T_i}(T_i)}$.

Calibration is carried out by modeling the forward variance. However, at the time of writing this project, I was under a substantial time constraint. Thus, I adopted following parameter estimates from Nonlinear Option Pricing by Guyon and Pierre-labordere.

$k_1 = 4, k_2 = 12.5\%, \rho_{X^1,X^2} = 0.3, \rho_{S,X_1} = -50\%, \rho_{S,X_2} = -50\%, \theta = 22.65\%\ \nu = 174\%$

Then, the deviation of volatility from the Dupire volatility will be adjusted by the particle methods during the simulation.

Following Guyon and Henry-Labordere (2013), we can generalize the Dupire equation to incorporate stochastic interest rates and SLV.

Claim 1[2].

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, S_t)\sqrt{\zeta_t^t}dW_t$$

Can be calibrated exactly to the market smile if and only if

$$\frac{\sigma(t,K)^2 \mathbb{E}\left[e^{-\int_0^t r(u)du}\,\zeta_t^t \,\middle|\, S_t = K\right]}{\mathbb{E}\left[e^{-\int_0^t r(u)du} \,\middle|\, S_t = K\right]} = \sigma_{Dupire}^2(t,K)^2 - \frac{\mathbb{E}\left[e^{-\int_0^t r(u)du}\,(r_t - f(o,t) \,\middle|\, S_t = K\right]}{\frac{1}{2}K\partial_k^2 C(t,K)}$$

Where $f(0,t)$ is the instantaneous forward rate $\frac{\partial lnP(0,t)}{\partial t}$ and $C(t,K)$ is the ordinary option price in the market.

To prove claim 1, if I denote $C_m(t,K)$ as the option price from a model, the dynamics of $C_m(t,K)$ should equal that of $C(t,K)$.

Let's define $P_t = e^{-\int_0^t r(u)du}\,(S_t - K)\mathbb{I}(S_t > K)$.

Then,

$$\frac{\partial P_t}{\partial t} = -r(t)P_t dt$$

$$\frac{\partial P_t}{\partial S_t} = e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K) + e^{-\int_0^t r(u)du}\,(S_t - K)\delta(S_t - K)$$

$$\underset{\substack{\uparrow \\ x\delta(x)=0 \text{ by dirac} \\ \text{delta properties}}}{=} e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K)$$

$$\frac{\partial^2 P_t}{\partial S_t^2} = e^{-\int_0^t r(u)du}\,\delta(S_t - K)$$

Thus, applying the Itô lemma,

$$dP_t = e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K)r_t K dt + e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K)S_t(r_t dt + \zeta_t^t \sigma(t,S_t)dW_t)$$
$$+ \frac{1}{2}K^2\zeta_t^t\sigma(t,K)^2 e^{-\int_0^t r(u)du}\,\delta(S_t - K)dt$$

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[2] Change of measure gives

$\frac{df_t}{f_t} = \sigma(t,S_t)\sqrt{\zeta_t^t}dW_t^T + \sigma_r B(t,T)dZ_t^T$ because a forward contract is martingale under $\mathbb{Q}^T$.

For a quanto, it would be $\frac{df_t}{f_t} = \left(r_f - r_d - \sigma_X\sigma(t,S_t)\right)dt + \sigma(t,S_t)\sqrt{\zeta_t^t}dW_t^T + \sigma_r B(t,T)dZ_t^T$

where r_f is the OIS rate in foreign currency, r_d is the OIS rate in domestic currency, and σ_X is the volatility of exchange rates. The drift arises from the quanto correction $r_f - \sigma_X\sigma(t,S_t)$.

Assuming we are working with a strict martingale rather than a local martingale,

$$\frac{dC_m(t,K)}{dt} = \mathbb{E}\left[e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K)\big(r_t - f(0,t)\big)K\right] + E\left[e^{-\int_0^t r(u)du}\,\mathbb{I}(S_t > K)\right]Kf(0,t) + \frac{1}{2}K^2\sigma^2(t,K)E\left[e^{-\int_0^t r(u)du}\,\zeta_t^t\delta(S_t - K)\right]$$

Assuming the interchange of differentiation and expectation is justified, for example by dominated convergence, and the partial derivatives, I have

$$\frac{\sigma(t,K)^2\mathbb{E}\left[e^{-\int_0^t r(u)du}\,\zeta_t^t\,\middle|\,S_t = K\right]}{\mathbb{E}\left[e^{-\int_0^t r(u)du}\,\middle|\,S_t = K\right]} = \sigma_{Dupire}^2(t,K)^2 - \frac{\mathbb{E}\left[e^{-\int_0^t r(u)du}\,(r_t - f(0,t))\,\middle|\,S_t = K\right]}{\frac{1}{2}K\partial_k^2 C(t,K)}$$

Everything should follow from the definition of the Duper local volatility. In a similar manner, we change the measure and obtain

$$\sigma(t,K)^2 = \left(\sigma_{Dupire}^2(t,K) - P(0,T)\frac{\mathbb{E}[P(t,T)^{-1}(r_t - f(0,t)\,|S_t = K]}{\frac{1}{2}K\partial_k^2 C(t,K)}\right)\frac{\mathbb{E}[P(t,T)^{-1}\,|S_t = K]}{\mathbb{E}[P(t,T)^{-1}\zeta_t^t\,|S_t = K]}$$

3. Simulation Procedure

Since for this project, we need to simulate LIBOR in addition to section 2.4. Therefore,

$$\left(W_{t_{k+1}}^S - W_{t_k}^S, \int_{t_k}^{t_{k+1}} e^{-a(t_{k+1}-u)}dZ_T(u), \int_{t_k}^{t_{k+1}} B(u,T)dZ_T(u), W_T(t_{k+1}) - W_T(t_k), W_{t_{k+1}}^1 - W_{t_k}^1, W_{t_{k+1}}^2 - W_{t_k}^2\right)$$

Which is a Gaussian vector with covariance matrix $\Sigma_k(t_{k+1} - t_k)$ with

$$\Sigma_k = \begin{pmatrix} 1 & \rho_{S,P}J_k^1 & \rho_{S,P}I_{t_k,t_{k+1},T}^2 & \rho_{S,L} & \rho_{s,X1} & \rho_{s,X_2} \\ \rho_{S,P}J_k^1 & J_k^2 & L_k & \rho_{L,F}J_k^1 & 0 & 0 \\ \rho_{S,P}I_{t_k,t_{k+1},T}^2 1 & L_k & I_{t_k,t_{k+1},T}^1 & \rho_{L,P}I_{t_k,t_{k+1},T}^2 & 0 & 0 \\ \rho_{S,L} & \rho_{L,F}J_k^1 2 & \rho_{L,P}I_{t_k,t_{k+1},T}^2 & 1 & 0 & 0 \\ \rho_{s,X_1} & 0 & 0 & 0 & 1 & \rho_{X_1,X_2} \\ \rho_{s,X_2} & 0 & 0 & 0 & \rho_{X_1,X_2} & 1 \end{pmatrix}$$

Where

$$J_k^1 = \frac{1 - e^{-a(t_{k+1}-t_k)}}{a(t_{k+1} - t_k)}$$

$$J_k^2 = \frac{1 - e^{-2a(t_{k+1}-t_k)}}{2a(t_{k+1} - t_k)}$$

$$L_k = \frac{J_k^1}{a} - e^{-a(T-t_{k+1})}\frac{e^{2at_{k+1}} - e^{2at_k}}{2a^2(t_{k+1} - t_k)}$$

For the rest of this section, for any variable ϕ, $\phi^{i,N}$ is defined as the i^{th} path from N simulation paths.

$$\sigma_N(t,S)^2 = \left(\sigma^2_{Dupire}(t,S) - P(0,T)\frac{1}{N}\left(P^{i,N}_{t,T}\right)^{-1}\left(r^{i,N}_t - f(0,t)\right)\mathbb{I}\left(S^{i,N}_t > S\right)\right) * \frac{\sum_{i=1}^{N}\left(P^{i,N}_{t,T}\right)^{-1}\delta_{t,N}\left(S^{i,N}_t - S\right)}{\left(P^{i,N}_{t,T}\right)^{-1}\left(\zeta^{t}_t{}^{i,N}\right)^2\delta_{t,N}\left(S^{i,N}_t - S\right)}$$

Where

$$\delta_{t,N}(x) = \frac{1}{h_{t,N}}K\left(\frac{x}{h_{t,N}}\right)$$

$$h_{t,N} = 1.5f(t,T)\sigma_{VS,t}\sqrt{\max\left(t,\frac{1}{4}\right)}N^{-\frac{1}{5}}$$

$$K(x) = \frac{15N}{16}(1-Nx^2)^2\mathbb{I}(|Nx| \le 1)$$

The choice of the kernel and $h_{t,N}$ are from Guyon and Henry-Labordere (2013).

$$r_{t_k+1} - f(0,t_{k+1}) = \left(r_{t_k} - f(0,t_k)\right)e^{-a(t_{k+1}-t_k)} + e^{-at_{k+1}}\int_{t_k}^{t_{k+1}}\sigma_r e^{au}dZ(u) + e^{-at_{k+1}}\int_0^{t_{k+1}}\sigma_r^2 e^{au}B(u,t_{k+1})du - e^{-at_{k+1}}\int_0^{t_k}\sigma_r^2 e^{au}B(u,t_k)du$$

where, from the discussion of the drift term in section 1.1, $dZ(u) = B(u,T)\sigma_r du + dZ_T(u)$

Therefore,

$$r_{t_k+1} - f(0,t_{k+1}) = \left(r_{t_k} - f(0,t_k)\right)e^{-a(t_{k+1}-t_k)} + \frac{\sigma_r^2}{a}B(t_k,t_{k+1})\left(1-\frac{1}{2}e^{-aT}\right) + a\sigma_r^2\left(I^2_{0,t_{k+1},t_{k+1}} - I^1_{0,t_{k+1},t_{k+1}}\right)t_{k+1} + \sigma_r^2 e^{-a(t_{k+1}-t_k)}\left(\frac{1}{2}B(0,2t_k) - B(0,t_k)\right) + e^{-at_{k+1}}\int_{t_k}^{t_{k+1}}\sigma_r e^{au}dZ_T(u)$$

Where,

$$I^1_{t_k,t_{k+1},T} = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}}B(u,T)^2du = \frac{a(t_{k+1}-t_k) - 2e^{-aT}(e^{at_{k+1}} - e^{at_k}) + \frac{1}{2}e^{-aT}(e^{2at_{k+1}} - e^{2at_k})}{a^3(t_{k+1}-t_k)}$$

$$I^2_{t_k,t_{k+1},T} = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}}B(u,T)du = \frac{a(t_{k+1}-t_k) - e^{-aT}(e^{at_{k+1}} - e^{at_k})}{a^2(t_{k+1}-t_k)}$$

And,

$$x(t_{k+1}) = r(t_{k+1}) - f(0,t_{k+1}) - a\sigma_r^2\left(I^2_{0,t_{k+1},t_{k+1}} - I^1_{0,t_{k+1},t_{k+1}}\right)t_{k+1}$$

$$P(t_{k+1},T) = P(t_k,T)\frac{P(0,t_k)}{P(0,t_{k+1})}\frac{A(0,t_{k+1})}{A(0,t_k)}\frac{A(t_{k+1},T)}{A(t_k,T)}e^{-\left(B(t_{k+1},T)x(t_{k+1})-B(t_k,T)x(t_k)\right)}$$

$$= P(t_k,T)\frac{P(0,t_k)}{P(0,t_{k+1})}e^{\frac{\sigma_r^2}{2}\left(-I^1_{t_k,t_{k+1},T}(t_{k+1}-t_k)+I^1_{0,t_{k+1},t_{k+1}}t_{k+1}-I^1_{0,t_k,t_k}t_k\right)}e^{-\left(B(t_{k+1},T)x(t_{k+1})-B(t_k,T)x(t_k)\right)}$$

Then, we have all the ingredients for the forward contract.

$$f_{t_{k+1}} = f_{t_k}\exp\Bigg(\sigma_N\left(t_k,\left(P_{t_k,T}\right)^{-1}f_{t_k}\right)\sqrt{\zeta^{t_k}_{t_k}}\left(W^S_T(t_{k+1})-W^S_T(t_k)\right)+\sigma_r\int_{t_k}^{t_{k+1}}B(u,T)dZ_T(u)$$
$$-\frac{1}{2}(t_{k+1}-t_k)\Bigg(\sigma_N^2\left(t_k,\left(P_{t_k,T}\right)^{-1}f_{t_k}\right)\zeta^{t_k}_{t_k}+\sigma_r^2I^1_{t_k,t_{k+1},T}(t_{k+1}-t_k)$$
$$+2\rho_{S,P}\sigma_r\sigma_N\left(t_k,\left(P_{t_k,T}\right)^{-1}f_{t_k}\right)\sqrt{\zeta^{t_k}_{t_k}I^2_{t_k,t_{k+1},T}(t_{k+1}-t_k)}\Bigg)\Bigg)$$

To obtain LIBOR rates for the barrier, I simulate, for $T_1 < i \le T$

$$\tilde{L}_i(t) = \tilde{L}_i(0)\exp\left(-\sum_{i<j\le T}\int_0^t\frac{\tau_j\tilde{L}_j(s)}{\left(1+\tau_j\tilde{L}_j(s)\right)}\sigma_j(s)\,ds-\frac{1}{2}\int_0^t\sigma_i(s)ds+\int_0^t\sigma_i(s)dW_T(s)\right)$$

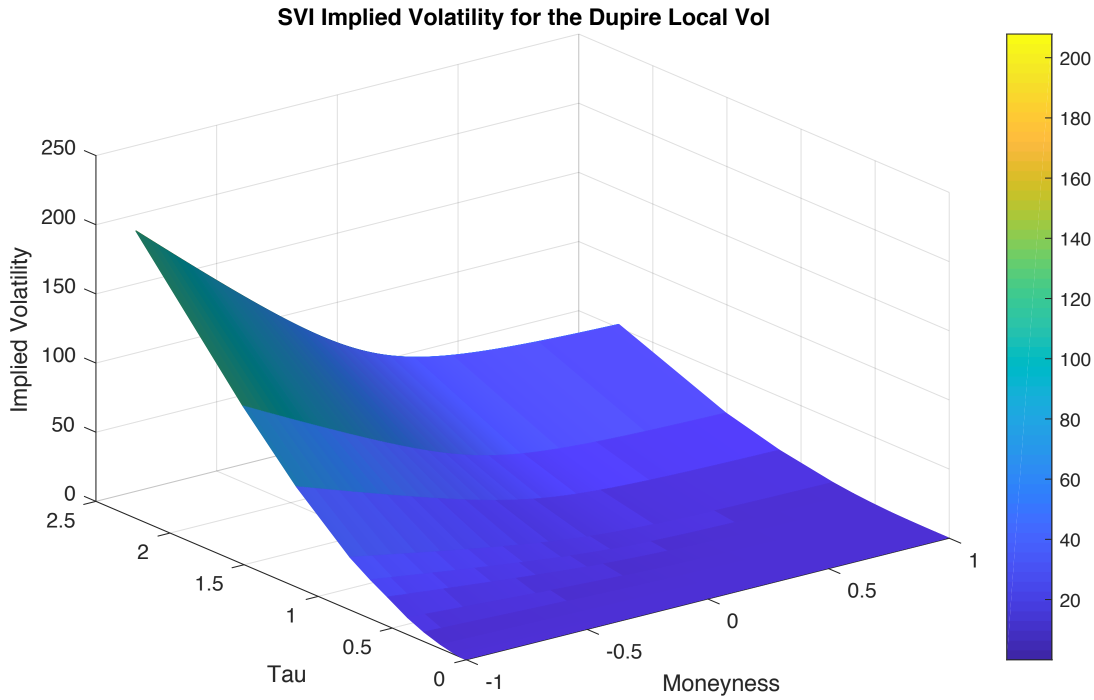
Then, based on the estimates, I obtain

$$\tilde{L}_{T_1}(t) = \tilde{L}_{T_1}(0)\exp\left(-\sum_{T_1<i\le T}\int_0^t\frac{\tau_i\tilde{L}_i(s)}{\left(1+\tau_i\tilde{L}_i(s)\right)}\sigma_i(s)\,ds-\frac{1}{2}\int_0^t\sigma_{T_1}(s)ds+\int_0^t\sigma_{T_1}(s)dW_T(s)\right)$$

Obtaining the price of the option then becomes the mean of the simulated payoffs multiplied by the discount factor.

4. Numerical Result

Time to Expiration	a	σ	m	b	ρ	RMSE
0.079	0.0001	0.3579	-0.0259	88.9779	-0.0028	0.1979
0.0357	0.0000	0.3723	-0.0255	59.9774	-0.0133	0.3079
0.0635	0.0000	0.2963	-0.0308	71.4296	-0.0188	0.3462
0.0913	0.0000	0.3128	-0.0303	61.0879	-0.0285	0.4459
0.1746	0.0000	0.3615	-0.0269	49.8549	-0.0631	0.6595
0.2857	0.0000	0.4145	-0.0238	41.8364	-0.1184	0.7438
0.3968	0.5223	0.4312	-0.0194	37.9090	-0.1711	0.7494
0.5317	2.6847	0.3900	-0.0200	35.2353	-0.2074	0.2732
0.6468	3.8026	0.3870	-0.0208	31.3483	-0.2503	0.6212
0.7857	5.0093	0.3853	-0.0209	27.9759	-0.3027	0.5454
1.1468	6.9670	0.3640	-0.0212	23.8577	-0.4174	0.2781
1.5079	8.1826	0.3565	-0.0216	20.8635	-0.5375	0.3256
2.2302	9.5053	0.3702	-0.0212	15.9427	-0.8256	0.2650



The logic of the computation is clear. To obtain the price of a particular option specification, we can use the Monte Carlo simulation.

5. Appendix

5.1. Elaborating notions introduced in the calibration of the Bergomi model

The Bergomi model attempts to capture the joint dynamics of the spot and its implied forward Variance Swap (VS) variances. To be more precise, Variance Swap (VS) volatility $\hat{\sigma}_{VS,T}(t)$ is defined as a constant such that the fair value of a variance with the following payoff is zero at the initial time t.

$$\text{VS payoff: } \hat{\sigma}^2{}_{VS,T}(t) - \frac{1}{T-t}\sum_{i=0}^{N-1} \ln^2\left(\frac{S_{(i+1)}}{S_i}\right)$$

Then,

$$\sum_{i=T_1}^{T_2-1} \ln^2\left(\frac{S_{(i+1)}}{S_i}\right) - \left(\hat{\sigma}^2{}_{VS,T_2}(t)(T_2-t) - \hat{\sigma}^2{}_{VS,T_1}(t)(T_1-t)\right)$$

Such that

$$\hat{\sigma}^2{}_{VS,T_1,T_2}(t) \equiv \frac{1}{T_2-T_1}\left(\hat{\sigma}^2{}_{VS,T_2}(t)(T_2-t) - \hat{\sigma}^2{}_{VS,T_1}(t)(T_1-t)\right)$$

Is called the discrete forward variance. Clearly, the continuous analog of the discrete forward variance is

$$\zeta_t^T \equiv \frac{d}{dT}\left((T-t)\hat{\sigma}^2{}_{VS,T}(t)\right)$$

In practice, VIX futures and VSTOXX futures have the expiration value of the 30-day variance swap volatility of the SP500 and STOXX50, respectively. Thus, VIX futures quotes at time 0 are used as proxies for ζ_0^T when S_t is the S&P500 Index based on $\hat{\sigma}^2{}_{VS,T,T+dt}(0)$.